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Coherent state path integrals in the Weyl representation

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Abstract

We construct a representation of the coherent state path integral using the Weyl symbol of the Hamiltonian operator. This representation is very different from the usual path integral forms suggested by Klauder and Skagerstam (1985 *Coherent States: Applications in Physics and Mathematical Physics* (Singapore: World Scientific)), which involve the normal or the antinormal ordering of the Hamiltonian. These different representations, although equivalent quantum mechanically, lead to different semiclassical limits. We show that the semiclassical limit of the coherent state propagator in the Weyl representation involves classical trajectories that are independent of the width of coherent states. This propagator is also free from the phase corrections found in Baranger *et al* (2001 *J. Phys. A: Math. Gen.* **34** 7227) for the two Klauder forms and provides an explicit connection between the Wigner and the Husimi representations of the evolution operator.

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1. Introduction

The set of coherent states forms a non-orthogonal overcomplete basis. This has important consequences for the path integral formulation of the propagator. It implies the existence of several forms of path integrals, all quantum mechanically equivalent, but each leading to a slightly different semiclassical limit. Klauder and Skagerstam (KS) [1] proposed two basic forms for the coherent state path integral, whose semiclassical limits were considered in [2]. It was shown in [2] that these two semiclassical propagators can be written in terms of classical complex trajectories, each governed by a different classical representation of the Hamiltonian operator $\hat{H}$: the P representation H_{P} in one case and the Q representation H_{Q} in the other. We briefly review these representations and their semiclassical limits in section 2. The two most important characteristics of these semiclassical formulae are, first, that the underlying classical dynamics depends explicitly on the width of coherent states. Second, the phase appearing in these semiclassical formulae is not just the action of the corresponding complex classical trajectory, but also contains a 'correction term' I that comes with different signs in each formula (see equations (15) and (16)).

In [2] it was also suggested that a semiclassical representation involving directly the Weyl representation of $\hat{H}$, or the *classical Hamiltonian* H_{W}, could probably be constructed and a formula for this representation was conjectured. The first attempt to derive such a formula was recently presented in [3]. The strategy used there was to build the propagator out of infinitesimal propagators that alternated between the two KS forms. The resulting semiclassical dynamics turned out to be governed by $(H_{\mathrm{Q}} + H_{\mathrm{P}})/2$, which coincides with the Weyl symbol for polynomial Hamiltonians with up to cubic terms in q and p only. The correction to the action was found to be $(I_{\mathrm{Q}} - I_{\mathrm{P}})/2$, which is also non-zero for general Hamiltonians. In this paper we construct a new representation of the quantum mechanical path integral in the coherent state representation that contains precisely H_{W} and derive its semiclassical limit. The new construction is based on the properties of translation and reflection operators [4, 5], which form the basis for expressing general operators. While in the KS path integrals each path contributes a term of the form $\exp \mathrm{i}S/\hbar$, where S is the action along the path (computed with either H_{Q} or H_{P}), the exponent in the new form is rather different and does not immediately resemble an action. Although the terms in this exponent can be rearranged so as to look similar to the action function, it is only when the limit of continuous paths is taken that one can really recognize the action as a part of the exponent.

We show that the semiclassical limit of the coherent state propagator in the Weyl representation is indeed given by the expression conjectured in [2]; the underlying dynamics is purely classical (independent of the width of the coherent states) and there is no correction term to be added to the action. More importantly, the new path integral representation allows for a direct connection between the coherent state representation of the evolution operator and its Weyl symbol.

The paper is organized as follows: in section 2 we review the path integral constructions of Klauder and Skagerstam and their semiclassical approximations. In section 3 we construct a new path integral representation and in section 4 we derive its semiclassical limit. The two path integrals of Klauder and Skagerstam are compared with the new form in section 5, where we also comment on the relevance of these results for numerical calculations. Finally, in section 6, we discuss the connection between the Weyl symbol of the evolution operator and the diagonal coherent state propagator.

2. The coherent state propagator and its semiclassical approximations

In this section we define the coherent state propagator and review the construction of the two path integrals suggested by Klauder and Skagerstam, showing how the symbols H_{Q} and H_{P} of the operator $\hat{H}$ appear in each of them. We also write down the semiclassical limit of these path integrals to compare with our results in the next section. Our presentation here is strongly based on [3].

2.1. The propagator

The coherent state $|z\rangle$ of a harmonic oscillator of mass m and frequency ω is defined by

$$|z\rangle = \mathrm{e}^{-\frac{1}{2}|z|^2} \mathrm{e}^{z\hat{a}^\dagger} |0\rangle, \tag{1}$$

with $|0\rangle$ the harmonic oscillator ground state and

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\hat{q}}{b} - \mathrm{i}\frac{\hat{p}}{c}\right), \qquad z = \frac{1}{\sqrt{2}}\left(\frac{q}{b} + \mathrm{i}\frac{p}{c}\right). \tag{2}$$

In the above $\hat{q}$, $\hat{p}$ and $\hat{a}^\dagger$ are operators; q and p are real numbers; and z is complex. The parameters $b = (\hbar/m\omega)^{\frac{1}{2}}$ and $c = (\hbar m\omega)^{\frac{1}{2}}$ define the length and momentum scales, respectively, and their product is $\hbar$.

For a time-independent Hamiltonian operator $\hat{H}$, the propagator in the coherent states representation is the matrix element of the evolution operator between the states $|z'\rangle$ and $|z''\rangle$:

$$K(z'', z', T) = \langle z''|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}T}|z'\rangle. \tag{3}$$

We restrict ourselves to Hamiltonians that can be expanded in a power series of the creation and annihilator operators $\hat{a}^\dagger$ and $\hat{a}$.

In the construction of a path integral for K, and also in the derivation of the semiclassical limit of the propagator, the Hamiltonian operator $\hat{H}$ is somehow replaced by a classical Hamiltonian function $H(q, p)$. This 'replacement', however, is not uniquely defined and the ambiguities that exist in the relation between the operator $\hat{H}$ and the function $H(q, p)$ also arise in connection with the overcompleteness of the coherent state basis, as we shall see in the next subsections.

There are actually many ways to associate a classical function of position and momentum $A(q, p)$ with a quantum mechanical operator $\hat{A}$ [6]. However, three of them are specially important. The first one, denoted by $A_{\mathrm{Q}}(q, p)$ and called the Q representation of the operator $\hat{A}$, is constructed as follows: one writes $\hat{A}$ in terms of the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ in such a way that all the creation operators appear to the left of the annihilation operators, making each monomial of $\hat{A}$ look like $c_{nm}\hat{a}^{\dagger n}\hat{a}^m$. Then we replace $\hat{a}$ by z and $\hat{a}^\dagger$ by $z^\star$. The inverse of this operation, which associates a quantum operator with a classical function, is called 'normal ordering'. In this case one first writes the classical function in terms of z and $z^\star$, with all the $z^\star$'s to the left of the z's, and then replace z by $\hat{a}$ and $z^\star$ by $\hat{a}^\dagger$.

The second possibility, called the P representation of $\hat{A}$, is obtained by a similar procedure, but this time the monomials of $\hat{A}$ are written in the opposite order, such that they look like $c_{nm}\hat{a}^n\hat{a}^{\dagger m}$. Once the operator has been put in this form one replaces again $\hat{a}$ by z and $\hat{a}^\dagger$ by $z^\star$ to obtain $A_{\mathrm{P}}(q, p)$. The inverse of this operation is called 'anti-normal ordering'. Note that the differences between the two representations come from the commutator of $\hat{q}$ and $\hat{p}$, which is proportional to $\hbar$. Therefore, these differences go to zero as $\hbar$ goes to zero.

There is, finally, a third representation which is the most symmetric of all, and therefore the most natural. It is given by the Wigner transformation

$$A_{\mathrm{W}}(q, p) = \int \mathrm{d}s\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}ps}\left\langle q - \frac{s}{2}\left|\hat{A}\right| q + \frac{s}{2}\right\rangle. \tag{4}$$

$A_{\mathrm{W}}(q, p)$ is called the Weyl representation of $\hat{A}$ [7, 5]. Its inverse transformation consists in writing the classical function in terms of z and $z^\star$ considering all possible orderings for each monomial and making a symmetric average between all possibilities before replacing z and $z^\star$ by the corresponding operators. As an illustration of these three representations we take

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2 + x^4$$

$(m = \hbar = 1)$, for which we obtain

$$H_{\mathrm{Q}} = \tfrac{1}{2}(p^2 + x^2) + x^4 + \tfrac{1}{4}(b^2 + b^{-2}) + 3b^2x^2 + 3b^4/4,$$
$$H_{\mathrm{P}} = \tfrac{1}{2}(p^2 + x^2) + x^4 - \tfrac{1}{4}(b^2 + b^{-2}) - 3b^2x^2 + 3b^4/4,$$
$$H_{\mathrm{W}} = \tfrac{1}{2}(p^2 + x^2) + x^4,$$

where b is the width of the coherent state. Note the term proportional to x^2 that appears with opposite signs in H_{Q} and H_{P} really modifying the classical dynamics with respect to H_{W}.

2.2. Basic path integrals and their semiclassical approximations

The calculation of the semiclassical propagator in the coherent state representation starting from path integrals was discussed in detail in [2]. In this section we summarize these previous results emphasizing the non-uniqueness of the semiclassical limit as a consequence of the overcompleteness of the coherent state representation. The reader is referred to [2] for details.

In order to write a path integral for $K(z'', T; z', 0)$, the time interval has to be divided into a large number of slices and, for each slice, an infinitesimal propagator has to be calculated. As pointed out by Klauder and Skagerstam [1, 8], there are at least two different ways to do that. Each of these gives rise to a different representation of the path integral. Although they correspond to identical quantum mechanical quantities, their semiclassical approximations are different. We review the construction of these two representations below.

The first form of path the integral is constructed by dividing the time interval T into N parts of size τ and inserting the unit operator

$$\mathbb{1} = \int |z\rangle \frac{\mathrm{d}z\,\mathrm{d}z^*}{2\pi\mathrm{i}} \langle z| \tag{5}$$

everywhere between adjacent propagation steps. We denote the real and imaginary parts of z and z^* by x and y, respectively. In all integrations, $\mathrm{d}z\,\mathrm{d}z^*/2\pi\mathrm{i}$ means $\mathrm{d}x\,\mathrm{d}y/\pi$. After the insertions, the propagator becomes a $2(N-1)$-fold integral over the whole phase space

$$K(z'', t; z', 0) = \int \left\{ \prod_{j=1}^{N-1} \frac{\mathrm{d}z_j\,\mathrm{d}z_j^*}{2\pi\mathrm{i}} \right\} \prod_{j=0}^{N-1} \left\{ \langle z_{j+1}|\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t_j)\tau}|z_j\rangle \right\}, \tag{6}$$

with $z_N = z''$ and $z_0 = z'$. Using the coherent state overlap formula

$$\langle z_{j+1}|z_j\rangle = \exp\left\{ -\frac{1}{2}|z_{j+1}|^2 + z^\star_{j+1}z_j - \frac{1}{2}|z_j|^2 \right\} \tag{7}$$

and expanding $\mathrm{e}^{-\mathrm{i}H\tau/\hbar} \approx 1 - \mathrm{i}H\tau/\hbar$ we write

$$\langle z_{j+1}|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t_j)\tau}|z_j\rangle = \exp\left\{ \frac{1}{2}(z^\star_{j+1} - z^\star_j)z_j - \frac{1}{2}z^\star_{j+1}(z_{j+1} - z_j) - \frac{\mathrm{i}\tau}{\hbar}\mathcal{H}_{j+1,j} \right\}, \tag{8}$$

where

$$\mathcal{H}_{j+1,j} \equiv \frac{\langle z_{j+1}|\hat{H}(t_j)|z_j\rangle}{\langle z_{j+1}|z_j\rangle} \equiv \mathcal{H}(z^\star_{j+1}, z_j; t_j) \tag{9}$$

and $(1 - \mathrm{i}\mathcal{H}_{j+1,j}\tau/\hbar)$ has been approximated again by $\mathrm{e}^{-\mathrm{i}\mathcal{H}_{j+1,j}\tau/\hbar}$. With these manipulations the first form of the propagator, which we shall call K_{Q}, becomes

$$K_{\mathrm{Q}}(z'', t; z', 0) = \int \left\{ \prod_{j=1}^{N-1} \frac{\mathrm{d}z_j\mathrm{d}z_j^*}{2\pi\mathrm{i}} \right\} \times \exp\left\{ \sum_{j=0}^{N-1} \left[\frac{1}{2}(z^\star_{j+1} - z^\star_j)z_j - \frac{1}{2}z^\star_{j+1}(z_{j+1} - z_j) - \frac{\mathrm{i}\tau}{\hbar}\mathcal{H}_{j+1,j} \right] \right\}. \tag{10}$$

When the limit $N \to \infty$ and $\tau \to 0$ is taken, the above summations turn into integrals. Also, $\mathcal{H}_{j+1,j}$ turns into the smooth Hamiltonian function $\mathcal{H}(z, z^\star) \equiv \langle z|\hat{H}|z\rangle$. Using the properties $\hat{a}|z\rangle = z|z\rangle$ and $\langle z|\hat{a}^\dagger = \langle z|z^\star$, we see that $\mathcal{H}$ can be easily calculated if $\hat{H}$ is written in terms of creation and annihilation operators with all $\hat{a}^\dagger$'s to the left of the $\hat{a}$'s. Therefore, $\mathcal{H}$ is exactly $H_Q(z, z^\star)$, the Q symbol of the Hamiltonian operator [7].

The second form of the path integral starts from the 'diagonal representation' of the Hamiltonian operator, namely

$$\hat{H} = \int |z\rangle h(z^\star, z) \frac{\mathrm{d}z\,\mathrm{d}z^*}{2\pi\mathrm{i}} \langle z|. \tag{11}$$

Assuming that $\hat{H}$ is either a polynomial in p and q or a converging sequence of such polynomials, this diagonal representation always exists. The calculation of h is not as direct as that of $\mathcal{H}$, but it can be shown [7] that $h(z^\star, z)$ is exactly H_P, the P symbol of $\hat{H}$. To facilitate the comparison between this form of path integral, which we call K_P and K_Q, it is convenient to divide the time interval T into $N-1$ intervals, rather than N. We write

$$K_P(z'', T; z', 0) = \langle z''| \prod_{j=1}^{N-1} \mathrm{e}^{-\frac{\mathrm{i}\tau}{\hbar}\hat{H}} |z'\rangle \tag{12}$$

and, following Klauder and Skagerstam, we write the infinitesimal propagators as

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}\tau} \approx \int |z_j\rangle \left(1 - \frac{\mathrm{i}\tau}{\hbar} h(z_j^\star, z_j)\right) \frac{\mathrm{d}z_j\,\mathrm{d}z_j^*}{2\pi\mathrm{i}} \langle z_j| \approx \int |z_j\rangle\, \mathrm{e}^{-\frac{\mathrm{i}\tau}{\hbar} h(z_j^\star, z_j)} \frac{\mathrm{d}z_j\,\mathrm{d}z_j^*}{2\pi\mathrm{i}} \langle z_j|. \tag{13}$$

The complete propagator K_P becomes

$$\begin{aligned} K_P(z_N, T; z_0, 0) &= \int \prod_{j=1}^{N-1} \frac{\mathrm{d}z_j \mathrm{d}z_j^*}{2\pi\mathrm{i}} \langle z_{j+1}|z_j\rangle \exp\left\{-\frac{\mathrm{i}\tau}{\hbar} h(z_j^\star, z_j)\right\} = \int \left\{\prod_{j=1}^{N-1} \frac{\mathrm{d}z_j \mathrm{d}z_j^*}{2\pi\mathrm{i}}\right\} \\ &\quad \times \exp\left\{\sum_{j=0}^{N-1} \left[\frac{1}{2}(z_{j+1}^\star - z_j^\star) z_j - \frac{1}{2} z_{j+1}^\star (z_{j+1} - z_j) - \frac{\mathrm{i}\tau}{\hbar} h(z_j^\star, z_j)\right]\right\}. \end{aligned} \tag{14}$$

Note that while the two arguments of H_Q in K_Q belong to two adjacent times in the mesh, the two arguments of H_P in K_P belong to the same time. Although both forms should give identical results when computed exactly, the differences between the two are important for the stationary exponent approximation, resulting in different semiclassical propagators. The semiclassical evaluation of K_Q and K_P was presented in detail in [2] (see also [9–11]). Here we only list the results:

$$K_Q(z'', t; z', 0) = \sum_\nu \sqrt{\frac{\mathrm{i}}{\hbar} \frac{\partial^2 S_{Q\nu}}{\partial u' \partial v''}} \exp\left\{\frac{\mathrm{i}}{\hbar}(S_{Q\nu} + I_{Q\nu}) - \frac{1}{2}(|z''|^2 + |z'|^2)\right\}, \tag{15}$$

$$K_P(z'', t; z', 0) = \sum_\nu \sqrt{\frac{\mathrm{i}}{\hbar} \frac{\partial^2 S_{P\nu}}{\partial u' \partial v''}} \exp\left\{\frac{\mathrm{i}}{\hbar}(S_{P\nu} - I_{P\nu}) - \frac{1}{2}(|z''|^2 + |z'|^2)\right\}, \tag{16}$$

where

$$S_{i\nu} = S_{i\nu}(v'', u', t) = \int_0^t \mathrm{d}t' \left[\frac{\mathrm{i}\hbar}{2}(\dot{u}v - \dot{v}u) - H_i(u, v, t')\right] - \frac{\mathrm{i}\hbar}{2}(u''v'' + u'v') \tag{17}$$

is the action and

$$I_i = \frac{1}{2} \int_0^T \frac{\partial^2 H_i}{\partial u \partial v} \mathrm{d}t \tag{18}$$

is a correction to the action. The index i assumes the values Q and P and sum over ν represents the sum over all 'contributing' (complex) classical trajectories satisfying Hamilton's equations

$$\mathrm{i}\hbar\dot{u} = +\frac{\partial H_i}{\partial v} \qquad \mathrm{i}\hbar\dot{v} = -\frac{\partial H_i}{\partial u}, \tag{19}$$

with boundary conditions

$$u(0) = z' \equiv u', \qquad v(t) = z''^{\star} \equiv v''. \tag{20}$$

The factors I_i are an important part of the formulae and they are absolutely necessary to recover the exact propagator for quadratic Hamiltonians. If one neglects them, even the harmonic oscillator comes out wrong. For a discussion about contributing and non-contributing trajectories, see [12, 13].

Finally we remember that the Weyl Hamiltonian can be obtained from $\hat{H}$ by completely symmetrizing the creation and annihilation operators. It turns out to be an exact average between H_{Q} and H_{P} if $\hat{H}$ contains up to cubic monomials in $\hat{a}$ and $\hat{a}^{\dagger}$, but only an approximate average for the other cases. The semiclassical formula with H_{Q} comes with a correction $+I_{\mathrm{Q}}$ and that with H_{P} comes with a correction of $-I_{\mathrm{P}}$. This suggests a third type of semiclassical approximation for the propagator, where one uses the Weyl Hamiltonian and no correction term, since the average of $+I_1$ and $-I_2$ should be approximately zero. This is the Weyl approximation, which was conjectured in [2]:

$$K_{\mathrm{W}}(z'', t; z', 0) = \sum_{\nu} \sqrt{\frac{\mathrm{i}}{\hbar}\frac{\partial^2 S_{\mathrm{W}}}{\partial u' \partial v''}} \exp\left\{\frac{\mathrm{i}}{\hbar} S_{\mathrm{W}} - \frac{1}{2}(|z''|^2 + |z'|^2)\right\}, \tag{21}$$

with S_{W} given by equation (17) with H_i replaced by H_{W}.

Of the three semiclassical approximations presented, the Weyl approximation seems to be the most natural, since it involves the classical Hamiltonian directly and no corrections to the action. However, this formula does not follow from the two most natural forms of the path integral proposed by Klauder and used in this section. In the next section we propose a third form of the path integral which is constructed directly in terms of H_{W} and whose semiclassical limit is indeed the formula above. For a direct comparison between these semiclassical formulae for short propagation times see [14].

3. Coherent state path integrals with the Weyl symbol

The new form of path integral we describe in this section is based on an expansion of the Hamiltonian in a continuous basis of reflection operators $\hat{R}_x$ whose coefficients $H(x)$ are exactly the Weyl symbol of $\hat{H}$. We first review the algebra of reflection and translation operators in quantum mechanics [4], following closely the presentation in [5]. We then use these results to construct the path integral.

3.1. Translation and reflection operators

Consider the family of translation operators

$$\hat{T}_{\xi} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p\hat{q} - q\hat{p})} = \mathrm{e}^{\mathrm{i}p\hat{q}/\hbar}\,\mathrm{e}^{-\mathrm{i}q\hat{p}/\hbar}\,\mathrm{e}^{-\mathrm{i}qp/2\hbar} = \mathrm{e}^{-\mathrm{i}q\hat{p}/\hbar}\,\mathrm{e}^{\mathrm{i}p\hat{q}/\hbar}\,\mathrm{e}^{+\mathrm{i}qp/2\hbar}, \tag{22}$$

where $\xi = (q, p)$ is a point in phase space. It can be shown that $\hat{T}_{\xi}$ form a complete basis, in the sense that any operator $\hat{A}$ can be expressed as

$$\hat{A} = \int \frac{\mathrm{d}\xi}{2\pi\hbar} A(\xi)\hat{T}_{\xi}. \tag{23}$$

The Fourier transform of the operators $\hat{T}_\xi$ forms a complementary family of *reflection* operators $\hat{R}_x$ which also form a basis:

$$\hat{R}_x = \frac{1}{4\pi\hbar}\int \mathrm{d}\xi\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(x\wedge\xi)}\hat{T}_\xi, \tag{24}$$

where $x = (\mathrm{Q}, \mathrm{P})$ and $x \wedge \xi = \mathrm{P}q - \mathrm{Q}p$. In terms of these operators we may write

$$\hat{A} = \int \frac{\mathrm{d}x}{\pi\hbar} A(x)\hat{R}_x = \frac{1}{4\pi^2\hbar^2}\int \mathrm{d}\xi\,\mathrm{d}x\, A(x)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(x\wedge\xi)}\hat{T}_\xi. \tag{25}$$

When this expression is inverted to write $A(x)$ in terms of $\hat{A}$, we find precisely the Weyl representation, as given by equation (4). This is shown in appendix A.

It is convenient to write some of these expressions in terms of $\hat{a}, \hat{a}^\dagger, z$ and z^* instead of $\hat{q}, \hat{p}, q$ and p. We find that

$$\hat{T}_\xi = \mathrm{e}^{(z\hat{a}^\dagger - z^*\hat{a})} = \mathrm{e}^{z\hat{a}^\dagger}\,\mathrm{e}^{-z^*\hat{a}}\,\mathrm{e}^{-|z|^2/2}, \tag{26}$$

which we recognize as the *displacement operator* [15–17] frequently used in quantum optics. Also

$$\langle z_k|\hat{T}_\xi|z_{k-1}\rangle = \exp(zz_k^* - z^*z_{k-1} - |z|^2/2)\langle z_k|z_{k-1}\rangle \tag{27}$$

and

$$\langle z_k|\hat{A}|z_{k-1}\rangle = \frac{1}{4\pi^2\hbar^2}\int \mathrm{d}x\, A(x)\langle z_k|z_{k-1}\rangle \\ \times \int \mathrm{d}\xi\, \exp\left(\frac{\mathrm{i}}{\hbar}(x\wedge\xi)\right)\exp(zz_k^* - z^*z_{k-1} - |z|^2/2). \tag{28}$$

Since the integral over $\xi = (q, p)$ is quadratic, it can be done immediately. Defining

$$w_k = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{Q}}{b} + \mathrm{i}\frac{\mathrm{P}}{c}\right) \tag{29}$$

(the index k is added for later convenience), we find

$$\langle z_k|\hat{A}|z_{k-1}\rangle = 2\int \frac{\mathrm{d}w_k\,\mathrm{d}w_k^*}{2\pi i} A(w_k, w_k^*) \\ \times \exp(-2|w_k|^2 + 2z_k^* w_k + 2z_{k-1}w_k^* - |z_k|^2/2 - |z_{k-1}|^2/2 - z_k^* z_{k-1}), \tag{30}$$

where

$$\frac{\mathrm{d}w_k\,\mathrm{d}w_k^*}{2\pi\mathrm{i}} = \frac{\mathrm{dQ}\,\mathrm{dP}}{2\pi\hbar}. \tag{31}$$

As the notation suggests, this expression will be our starting point to construct the path integral. When $\hat{A}$ is replaced by the infinitesimal propagator $\mathrm{e}^{-\mathrm{i}\hat{H}\tau/\hbar} \approx 1 - \mathrm{i}\hat{H}\tau/\hbar$ and a sequence of these matrix elements are multiplied together, we will find that all the z_k's and z_k^*'s appear only in quadratic forms and can be integrated over. The resulting path integral will be written in the new variables w.

3.2. The path integral

We start from

$$K(z'', t; z', 0) = \int \left\{\prod_{j=1}^{N-1}\frac{\mathrm{d}z_j\,\mathrm{d}z_j^*}{2\pi\mathrm{i}}\right\}\prod_{j=1}^{N}\left\{\langle z_j|\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t_j)\tau}|z_{j-1}\rangle\right\}, \tag{32}$$

where $z_N = z''$, $z_0 = z'$, τ is the time step, $N\tau = T$ and we take N to be even for convenience. The infinitesimal propagators can be calculated with equation (30) by simply replacing $A(w_k, w_k^*)$ by $\mathrm{e}^{-\mathrm{i}H(x_k)\tau/\hbar}$ where $H(x_k)$ is the Weyl symbol of $\hat{H}$ calculated at $(\mathrm{Q}_k, \mathrm{P}_k)$. We obtain

$$K(z'', t; z', 0) = 2^N \int \left\{ \prod_{j=1}^{N} \frac{\mathrm{d}w_j\,\mathrm{d}w_j^*}{2\pi\mathrm{i}} \right\} \int \left\{ \prod_{j=1}^{N-1} \frac{\mathrm{d}z_j\,\mathrm{d}z_j^*}{2\pi\mathrm{i}} \right\}$$
$$\times \exp\left\{ \sum_{k=1}^{N} \left[-\frac{\mathrm{i}}{\hbar} H_k \tau - 2|w_k|^2 + 2z_k^* w_k + 2z_{k-1} w_k^* - \frac{|z_k|^2}{2} - \frac{|z_{k-1}|^2}{2} - z_k^* z_{k-1} \right] \right\}, \tag{33}$$

where $H_k = H(w_k, w_k^*)$. The integrals over the z_j's and the z_j^*'s can be performed exactly. When this is done we find

$$K(z'', t; z', 0) = \int \left\{ \prod_{j=1}^{N} \frac{\mathrm{d}w_j \mathrm{d}w_j^*}{\pi\mathrm{i}} \right\} \exp\left(\phi_N - \frac{|z'|^2}{2} - \frac{|z''|^2}{2} \right) = \int \mathcal{D}[w, w^*]$$
$$\times \exp\left(\psi[w, w^*] + 2C[w, w^*]z''^* - 2C^*[w, w^*]z' - \frac{|z'|^2}{2} - \frac{|z''|^2}{2} + z'z''^* \right), \tag{34}$$

where

$$\phi_N = \sum_{k=1}^{N} \left[-\mathrm{i}\tau H_k/\hbar - 2|w_k|^2 + 2z''^* w_{N+1-k}(-1)^{k+1} + 2z' w_k^* (-1)^{k+1} \right]$$
$$+ 4\sum_{k=1}^{N-1} \sum_{j=1}^{k} w_{k+1}^* w_{k+1-j} (-1)^{j+1} + z'z''^*. \tag{35}$$

In the second line of (34) we have written the dependence of the propagator on z' and z''^* explicitly and defined

$$\begin{aligned} \psi_N &= \sum_{k=1}^{N} [-\mathrm{i}\tau H_k/\hbar - 2|w_k|^2] + 4\sum_{k=1}^{N-1} \sum_{j=1}^{k} w_{k+1}^* w_{k+1-j} (-1)^{j+1}, \\ C_N &= \sum_{k=1}^{N} w_{N+1-k} (-1)^{k+1}. \end{aligned} \tag{36}$$

3.3. Alternative form and the limit of continuum

Equations (34) and (35) correspond to the coherent state path integral in the Weyl representation. It is very different from the previous forms presented in section 2 in two respects: the measure lacks a factor 2 in the denominator and, more importantly, the exponent does not resemble an action at all. We shall comment more about this particular form of the action in section 6, when we compare this result with the path integral for the Weyl representation of the evolution operator. Although these expressions seem to be the most practical for actual calculations, we can manipulate the terms in ϕ_N to make it look more familiar and similar to an action function. However, it is only when we take the limit of the continuum that we really recognize the action as part of the exponent. We shall do these manipulations now, but we insist that equations (34) and (35) are the direct analogues of equations (10) and (14) for the Q and P representations respectively. Although unusual, and perhaps more complicated, we shall see that, in the semiclassical limit, the Weyl form becomes the simplest of them all.

We show in appendix B that the quadratic terms in ϕ_N can be written as

$$\begin{aligned} -\sum_{k=1}^{N} 2|w_k|^2 + 4\sum_{k=1}^{N-1}\sum_{j=1}^{k} w_{k+1}^* w_{k+1-j}(-1)^{j+1} \\ = 2\sum_{k=1,3}^{N-1}[w_k(w_{k+1}^* - w_k^*) - w_{k+1}^*(w_{k+1} - w_k)] \\ -4\sum_{k=1,3}^{N-1}(w_{k+1} - w_k)\sum_{l=k+1,k+3}^{N-2}(w_{l+2}^* - w_{l+1}^*), \end{aligned} \tag{37}$$

where the sums on the right-hand side go in steps of two. The terms proportional to z' and z''^* can also be rewritten as

$$\begin{aligned} \sum_{k=1}^{N} w_k^*(-1)^{k+1} &= -\sum_{k=1,3}^{N-1}(w_{k+1}^* - w_k^*) \\ \sum_{k=1}^{N} w_{N+1-k}(-1)^{k+1} &= \sum_{k=1,3}^{N-1}(w_{k+1} - w_k). \end{aligned} \tag{38}$$

When these terms are replaced in the exponent we get

$$\begin{aligned} \phi_N = 2\sum_{k=1,3}^{N-1}[w_k(w_{k+1}^* - w_k^*) - w_{k+1}^*(w_{k+1} - w_k)] - \frac{\mathrm{i}\tau}{\hbar}\sum_{k=1}^{N} H_k \\ - 4\sum_{k=1,3}^{N-1}(w_{k+1} - w_k)\sum_{l=k+1,k+3}^{N-2}(w_{l+2}^* - w_{l+1}^*) \\ - 2z'\sum_{k=1,3}^{N-1}(w_{k+1}^* - w_k^*) + 2z''^*\sum_{k=1,3}^{N-1}(w_{k+1} - w_k) + z'z''^*. \end{aligned} \tag{39}$$

This is the alternative discrete version of ϕ_N. Although not much enlightening than the original form, equation (35), the first line shows a closer resemblance to the usual action function. More importantly, this expression is ready for the continuum limit. Taking $N \to \infty$, $\tau \to 0$ with $N\tau = T$ we obtain

$$\begin{aligned} \phi = -\frac{\mathrm{i}\tau}{\hbar}\int_0^T H\,\mathrm{d}t + \int_0^T (w\dot{w}^* - w^*\dot{w})\,\mathrm{d}t - \int_0^T \dot{w}(t)\int_t^T \dot{w}^*(t')\,\mathrm{d}t'\,\mathrm{d}t \\ - z'\int_0^T \dot{w}^*\,\mathrm{d}t + z''^*\int_0^T \dot{w}\,\mathrm{d}t + z'z''^*. \end{aligned} \tag{40}$$

Note that the factors of 2 and 4 compensate for the sums in steps of two.

The integrals in the last term on the first line can be rewritten as

$$\int_0^T \dot{w}(t)[w^*(T) - w^*(t)]\,\mathrm{d}t = w^*(T)[w(T) - w(0)] - \int_0^T \dot{w}(t)w^*(t)\,\mathrm{d}t. \tag{41}$$

The last term above cancels one of the terms in equation (40). After performing the integrals on the second line of equation (40), making some simple rearrangements and an integration by parts, we can write the exponent in the form

$$\phi = \frac{\mathrm{i}}{\hbar}S + (z' - w(0))\left[w^*(0) + \frac{z''^* - w^*(T)}{2}\right] + (z''^* - w^*(T))\left[w(T) + \frac{z' - w(0)}{2}\right], \tag{42}$$

where S is the (complex) action [2]

$$S = \int_0^T \left[\frac{\mathrm{i}\hbar}{2}(w^*\dot{w} - w\dot{w}^*) - H\right]\mathrm{d}t - \frac{\mathrm{i}\hbar}{2}(w^*(T)w(T) + w^*(0)w(0)). \quad (43)$$

Note that the action in the coherent state representation is not just the integral corresponding to $p\dot{q} - H$, but it also includes important boundary terms. Besides, the exponent ϕ of the path integral is not just the action but it also includes further boundary terms. We shall see, however, that the extra terms in equation (42) vanish in the semiclassical limit.

4. Semiclassical limit

The semiclassical limit of the propagator is obtained by performing the integrals over w_k and w_k^* with the stationary phase approximation. Because the exponent ϕ_N is not a phase, but a complex quantity, we use the terminology 'stationary exponent approximation'.

4.1. The stationary exponent condition

Using equation (35) for N even and $l \neq 1$ even we obtain

$$\frac{\partial\phi_N}{\partial w_l^*} = -\frac{\mathrm{i}\tau}{\hbar}\frac{\partial H_l}{\partial w_l^*} - 2w_l - 2z' + 4[w_{l-1} - w_{l-2} + \cdots - w_2 + w_1] \equiv 0$$

and

$$\frac{\partial\phi_N}{\partial w_{l+1}^*} = -\frac{\mathrm{i}\tau}{\hbar}\frac{\partial H_{l+1}}{\partial w_{l+1}^*} - 2w_{l+1} + 2z' + 4[w_l - w_{l-1} + \cdots + w_2 - w_1] \equiv 0.$$

Adding these two equations we obtain simply

$$-\frac{\mathrm{i}}{\hbar}\frac{1}{2}\left[\frac{\partial H_l}{\partial w_l^*} + \frac{\partial H_{l+1}}{\partial w_{l+1}^*}\right] = \frac{w_{l+1} - w_l}{\tau}. \quad (44)$$

For $l = 1$ we get

$$\frac{\partial\phi_N}{\partial w_1^*} = -\frac{\mathrm{i}\tau}{\hbar}\frac{\partial H_1}{\partial w_1^*} - 2w_1 + 2z' \equiv 0. \quad (45)$$

For the derivatives with respect to w_l we proceed in the same way. For l odd we get

$$\frac{\partial\phi_N}{\partial w_l} = -\frac{\mathrm{i}\tau}{\hbar}\frac{\partial H_l}{\partial w_l} - 2w_l^* - 2z''^* + 4[w_{l+1}^* - w_{l+2}^* + \cdots - w_{N-1}^* + w_N^*] \equiv 0$$

and

$$\frac{\partial\phi_N}{\partial w_{l+1}} = -\frac{\mathrm{i}\tau}{\hbar}\frac{\partial H_{l+1}}{\partial w_{l+1}} - 2w_{l+1}^* + 2z''^* + 4[w_{l+2}^* - w_{l+1}^* + \cdots + w_{N-1}^* - w_N^*] \equiv 0.$$

Adding the two equations we obtain

$$-\frac{\mathrm{i}}{\hbar}\frac{1}{2}\left[\frac{\partial H_l}{\partial w_l} + \frac{\partial H_{l+1}}{\partial w_{l+1}}\right] = -\frac{w_{l+1}^* - w_l^*}{\tau}. \quad (46)$$

Finally for $l = N$ we get

$$\frac{\partial\phi_N}{\partial w_N} = -\frac{\mathrm{i}\tau}{\hbar}\frac{\partial H_1}{\partial w_N} - 2w_N^* + 2z''^* \equiv 0. \quad (47)$$

Taking the continuum limit and using the u and v variables in place of w and w^*, equations (44), (46), (45) and (47) become

$$\mathrm{i}\hbar\dot{u} = +\frac{\partial H_W}{\partial v}, \qquad \mathrm{i}\hbar\dot{v} = -\frac{\partial H_W}{\partial u}, \quad (48)$$

with boundary conditions

$$u(0) = z', \qquad v(T) = z''^{\star}. \tag{49}$$

The average of the derivatives at consecutive time steps that appear on the left-hand side of equations (44) and (46) resembles the stationary conditions obtained in [3]. In that case, however, one of the derivatives involved H_{P} and the other H_{Q}.

4.2. Expansion around the stationary trajectory

Let w_k^0 and w_k^{*0} represent the stationary trajectory and $w_k^0 + \xi_k$ and $w_k^{*0} + \xi_k^*$ a nearby path. Expanding the exponent up to second order around the stationary trajectory we get

$$\phi_N = \phi_N^0 + \delta^2\phi_N + O(3) \tag{50}$$

(the first-order term is zero) with

$$\begin{aligned}\delta^2\phi_N &= \sum_{k=1}^{N}\left\{-\frac{\mathrm{i}\tau}{2\hbar}\left[A_k\xi_k^2 + 2C_k\xi_k\xi_k^* + B_k\xi_k^{*2}\right] - 2\xi_k\xi_k^*\right\} + 4\sum_{k=1}^{N-1}\xi_{k+1}^*\sum_{j=1}^{k}\xi_{k+1-j}(-1)^{j+1}\\ &\equiv -\frac{1}{2}X^T\tilde{\Delta}_N X,\end{aligned} \tag{51}$$

where $X^T = (\xi_N, \xi_N^*, \xi_{N-1}, \ldots, \xi_1, \xi_1^*)$ and

$$A_k = \frac{\partial^2 H_k}{\partial w_k^2}, \qquad B_k = \frac{\partial^2 H_k}{\partial w_k^{*2}}, \qquad C_k = \frac{\partial^2 H_k}{\partial w_k \partial w_k^*} \tag{52}$$

are calculated at the stationary trajectory.

When the limit of the continuum is taken, the boundary conditions (49) kill the extra terms in the exponent ϕ, equation (42), which becomes simply the action of the complex trajectory. Therefore the semiclassical propagator becomes

$$K_{\mathrm{W}}(z', z'', T) = \exp\left(\frac{\mathrm{i}}{\hbar}S_{\mathrm{W}} - \frac{1}{2}(|z'|^2 + |z''|^2)\right)\lim_{N\to\infty}\frac{2^N}{\sqrt{(-1)^N\det(\tilde{\Delta}_N)}}. \tag{53}$$

As usual, the calculation of the determinant of the quadratic form is the most lengthy step of the semiclassical calculation. In this case the calculation is particularly tricky, because of the double sum in the last term of the first line of equation (51). To avoid losing the focus with this lengthy algebra here we do the calculation in appendix C. The final result is indeed the conjectured formula, equation (21), that we repeat here:

$$K_{\mathrm{W}}(z'', t; z', 0) = \sqrt{\frac{\mathrm{i}}{\hbar}\frac{\partial^2 S_{\mathrm{W}}}{\partial u'\partial v''}}\exp\left\{\frac{\mathrm{i}}{\hbar}S_{\mathrm{W}} - \frac{1}{2}(|z''|^2 + |z'|^2)\right\}. \tag{54}$$

Of course, if there is more than one stationary trajectory, one should sum over all the contributing ones.

5. A comparison between the three forms of the path integral

In principle, all discrete forms of path integrals given by equations (10), (14) and (34) are quantum mechanically equivalent. For fixed N, however, they are not identical and in the limit $N \to \infty$ there are well-known convergence problems, making the comparison difficult. In order to illustrate the differences between the three forms we shall study the discrete propagators for a simple harmonic oscillator. The Hamiltonian operator is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2 x^2}{2} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

and, choosing the coherent state width as $b = \sqrt{\hbar/m\omega}$, the classical symbols, in the u and v variables, are

$$H_{\mathrm{Q}} = \hbar\omega\left(uv + \tfrac{1}{2}\right) \qquad H_{\mathrm{P}} = \hbar\omega\left(uv - \tfrac{1}{2}\right) \qquad H_{\mathrm{W}} = \hbar\omega uv.$$

Using H_{W} in the stationary conditions (44)–(47) we obtain the stationary path

$$w_k = \frac{\alpha^{*k-1}}{\alpha^k} z', \qquad w_k^* = \frac{\alpha^{*N-k}}{\alpha^{N-k+1}} z''^*,$$

where

$$\alpha \equiv 1 + \mathrm{i}\tau\omega/2.$$

The calculation of the phase ϕ_N^0 at the stationary trajectory is lengthy but involves only simple geometric sums. Several simplifications occur when all the terms in ϕ_N^0 are added together and the result is

$$\phi_N^0 = \left(\frac{\alpha^*}{\alpha}\right)^N z'z''^* - \frac{1}{2}(|z'|^2 + |z''|^2).$$

The determinant of the quadratic form is calculated in appendix C and results in (see equations (C1) and (C6))

$$\det \tilde{\Delta}_N = 2^{2N}\mathrm{i}^{2N}\alpha^{2N}. \tag{55}$$

Putting everything together we obtain

$$K_{\mathrm{W}}(z'', z', T) = (1 + \mathrm{i}\tau\omega/2)^{-N} \exp\left(\left(\frac{1 - \mathrm{i}\tau\omega/2}{1 + \mathrm{i}\tau\omega/2}\right)^N z'z''^* - |z'|^2/2 - |z''|^2/2\right), \tag{56}$$

which clearly converges to the exact propagator as $\tau \to 0$. Doing similar calculations for the Q and P propagators we find

$$K_{\mathrm{Q}}(z'', z', T) = \exp(-\mathrm{i}\omega T/2 + (1 - \mathrm{i}\tau\omega)^N z'z''^* - |z'|^2/2 - |z''|^2/2) \tag{57}$$

and

$$K_{\mathrm{P}}(z'', z', T) = (1 + \mathrm{i}\tau\omega)^{-N} \exp\left(\mathrm{i}\omega T/2 + \frac{z'z''^*}{(1 + \mathrm{i}\tau\omega)^N} - |z'|^2/2 - |z''|^2/2\right), \tag{58}$$

which also converge to the exact result. Note that the overall phase $-\mathrm{i}\omega T/2$ comes out exact for K_{Q} even in the discrete form. However, the term multiplying $z'z''^*$, which goes to $\mathrm{e}^{-\mathrm{i}\omega T}$ as $N \to \infty$, converges much faster for K_{W} than the corresponding terms in K_{Q} or K_{P}. Moreover, for any finite value of N, this term has unit modulus in K_{W}, while its modulus is larger than 1 for K_{Q} and smaller than 1 in K_{P}. Just for the sake of comparison let us call this coefficient μ. Taking $\omega T = 2\pi$ and $N = 100$ we find $\mu_{\mathrm{Q}} \approx 1.22 + 0.01\mathrm{i}$, $\mu_{\mathrm{P}} \approx 0.82 + 0.007\mathrm{i}$ and $\mu_{\mathrm{W}} \approx 0.999998 + 0.002\mathrm{i}$. This suggests that the new path integral representation should be better than the two KS forms for numerical evaluations.

6. Connecting the Wigner and the Husimi propagators

In this section we show that the Weyl representation of the evolution operator

$$U(q, p, T) = \int \langle q - s/2|\, \mathrm{e}^{-\mathrm{i}\hat{H}T/\hbar} |q + s/2\rangle\, \mathrm{e}^{\mathrm{i}ps/\hbar}\, \mathrm{d}s \tag{59}$$

can be directly related to the path integral representation derived in section 3. This is an interesting formal result that was also obtained by Ozorio de Almeida in section 6 of [5] starting from the opposite direction, i.e. from the path integral representation of U. The result provides an explicit connection between these two famous phase space representations of quantum mechanics. As we shall see, the connection is very simple when written in terms of path integrals.

We start by rewriting equation (59) as

$$\begin{aligned} U(q,p,T) &= \int \langle q-s/2|z''\rangle\langle z''|\hat{U}|z'\rangle\langle z'|q+s/2\rangle \mathrm{e}^{\mathrm{i}ps/\hbar}\,\mathrm{d}s\,\frac{\mathrm{d}z'\,\mathrm{d}z'^*}{2\pi\mathrm{i}}\frac{\mathrm{d}z''\,\mathrm{d}z''^*}{2\pi\mathrm{i}} \\ &= \int \mathcal{D}[w,w^*]\,\mathrm{e}^{\psi}\int \mathrm{d}s\,\mathrm{e}^{\mathrm{i}ps/\hbar}\int \frac{\mathrm{d}z'\,\mathrm{d}z'^*}{2\pi\mathrm{i}}\frac{\mathrm{d}z''\,\mathrm{d}z''^*}{2\pi\mathrm{i}}\langle\beta|z''\rangle\langle z'|\alpha\rangle \\ &\quad\times\exp\left[-\frac{|z'|^2}{2}-\frac{|z''|^2}{2}+z'z''^*+2Cz''^*-2C^*z'\right], \end{aligned} \tag{60}$$

where we used equations (34) and (36) and defined $\alpha = q+s/2$ and $\beta = q-s/2$ in the second line. The integrals in z' and z'' are quadratic and can be performed analytically. The integral over z' is straightforward and gives

$$\begin{aligned} U(q,p,T) &= \frac{1}{\pi^{1/4}b^{1/2}}\int \mathcal{D}[w,w^*]\,\mathrm{e}^{\psi}\int \mathrm{d}s\,\mathrm{e}^{\mathrm{i}ps/\hbar}\int \frac{\mathrm{d}z''\,\mathrm{d}z''^*}{2\pi\mathrm{i}}\langle\beta|z''\rangle \\ &\quad\times\exp\left[-\frac{|z''|^2}{2}+2Cz''^*-\frac{\alpha^2}{2b^2}-\frac{z''^*-2C^*}{2}+\frac{\alpha\sqrt{2}}{b}(z''^*-2C^*)\right]. \end{aligned} \tag{61}$$

It can be seen by inspection that the exponent in the second line above can be written as

$$\pi^{1/4}b^{1/2}\langle z''|\alpha+A\rangle\,\mathrm{e}^{-B}, \tag{62}$$

with $A = b\sqrt{2}(C+C^*)$ and $B = -A^2/2b^2 - A\alpha/b^2 + 2C^{*2} + 2\sqrt{2}\alpha C^*/b$. When (62) is substituted into (61) the integral in z'' produces $\langle\beta|\alpha+A\rangle = \delta(\alpha-\beta+A) = \delta(s+A)$. The delta function takes care of the integral over s and after some simplifications we obtain simply

$$U(q,p,T) = \int \mathcal{D}[w,w^*]\exp(\psi + 2Cz^* - 2C^*z + 2|C|^2), \tag{63}$$

where $z = (q/b + \mathrm{i}pb/\hbar)/\sqrt{2}$. A comparison with equation (34) shows that the only difference between the path integrals for $U(q,p,T)$ and $K(z,z,T)$ is the extra term $2|C|^2$, which promotes the 'unsmoothing' of the coherent state propagator. Conversely, the diagonal coherent state propagator has the extra term $-2|C|^2$ with respect to U, smoothing it out. The coefficient C can actually be interpreted as the Wigner chord linking the ends of a polygon in phase space whose sides are centred on $(\mathrm{Q}_k, \mathrm{P}_k)$ defined by $w_k = (\mathrm{Q}_k/b + \mathrm{i}b\mathrm{P}_k/\hbar)/\sqrt{2}$. Similarly, the complete exponent in equation (63) can be identified with the action for the polygonal path with endpoints centred in (q,p) and whose sides are centred on $(\mathrm{Q}_k, \mathrm{P}_k)$ [5]. Finally, we can calculate explicitly the two terms that involve q and p in (63). Using the definition of C in equation (36) we find that

$$2Cz^* - 2C^*z = \sum_{k=1}^{N}\frac{2\mathrm{i}}{\hbar}(\mathrm{Q}_k p - \mathrm{P}_k q), \tag{64}$$

which is the sum of the symplectic areas between $X_k = (\mathrm{Q}_k, \mathrm{P}_k)$ and $x = (q,p)$, and is independent of the width b.

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Appendix A. Expansion in reflection and translation operators

This appendix follows closely the demonstration in [5]. A comparison between equations (23) and (25) shows that

$$A(\xi) = \frac{1}{2\pi\hbar}\int \mathrm{d}x\, A(x)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}x\wedge\xi} \tag{A1}$$

and, inverting the Fourier transform,

$$A(x) = \frac{1}{2\pi\hbar}\int \mathrm{d}\xi\, A(\xi)\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}x\wedge\xi}. \tag{A2}$$

Using equation (23) again in the coordinate representation we obtain

$$\begin{aligned}\langle q_+|\hat{A}|q_-\rangle &= \int \frac{\mathrm{d}\xi}{2\pi\hbar}A(\xi)\langle q_+|\hat{T}_\xi|q_-\rangle \\ &= \int \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi\hbar}A(q,p)\delta(q_+ - q_- - q)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(q_-+\frac{q}{2})} \\ &= \int \frac{\mathrm{d}p}{2\pi\hbar}A(q_+ - q_-, p)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{q_+ + q_-}{2}p}.\end{aligned} \tag{A3}$$

This Fourier transform can be inverted as follows: we define $q' = q_+ - q_-$, $\bar{\mathrm{Q}} = (q_+ + q_-)/2$, multiply both sides by $\mathrm{e}^{-\mathrm{i}p'\bar{\mathrm{Q}}/\hbar}$ and integrate over $\bar{\mathrm{Q}}$. The integral over $\bar{\mathrm{Q}}$ on the right-hand side yields a delta function on $p - p'$ and we obtain

$$A(\xi) = \int \mathrm{d}\bar{Q}\langle\bar{Q} + q/2|\hat{A}|\bar{Q} - q/2\rangle\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\bar{\mathrm{Q}}}, \tag{A4}$$

where (q', p') has been changed back to (q, p). Finally we use equation (A2) to get $A(x)$:

$$\begin{aligned}A(x) &= \frac{1}{2\pi\hbar}\int \mathrm{d}q\,\mathrm{d}p\,\mathrm{d}\bar{\mathrm{Q}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(\mathrm{Q}-\bar{\mathrm{Q}})-\frac{\mathrm{i}}{\hbar}\mathrm{P}q}\langle\bar{\mathrm{Q}} + q/2|\hat{A}|\bar{\mathrm{Q}} - q/2\rangle \\ &= \int \mathrm{d}q\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathrm{P}q}\langle\mathrm{Q} + q/2|\hat{A}|\mathrm{Q} - q/2\rangle,\end{aligned} \tag{A5}$$

which is the same as equation (4).

Appendix B. Proof of equation (37)

First we rewrite, for N even,

$$\begin{aligned}4\sum_{k=1}^{N-1}\sum_{j=1}^{k} w_{k+1}^* w_{k+1-j}(-1)^{j+1} &= 4w_2^*w_1 + 4w_3^*[w_2 - w_1] + 4w_4^*[w_3 - (w_2 - w_1)] \\ &\quad + 4w_5^*[(w_4 - w_3) + (w_2 - w_1)] + 4w_6^*[w_5 - (w_4 - w_3) - (w_2 - w_1)] \\ &\quad \vdots \\ &\quad + 4w_N^*[w_{N-1} - (w_{N-2} - w_{N-3}) - \cdots - (w_2 - w_1)] \\ &= 4[w_2^*w_1 + w_4^*w_3 + w_6^*w_5 + \cdots + w_N^*w_{N-1}] - 4(w_2 - w_1)[(w_4^* - w_3^*) \\ &\quad + (w_6^* - w_5^*) + \cdots + (w_N^* - w_{N-1}^*)] - 4(w_4 - w_3)[(w_6^* - w_5^*) \\ &\quad + (w_8^* - w_7^*) + \cdots + (w_N^* - w_{N-1}^*)] \\ &\quad \vdots \\ &\quad - 4(w_{N-2} - w_{N-3})[w_N^* - w_{N-1}^*] \\ &= 4\sum_{k=1,3}^{N-1} w_{k+1}^*w_k - 4\sum_{k=1,3}^{N-3}(w_{k+1} - w_k)\sum_{l=k+1,k+3}^{N-2}(w_{l+2}^* - w_{l+1}^*).\end{aligned} \tag{B1}$$

The second term is already in the form needed for equation (37). The first term is now modified as follows: half of it remains unchanged and, to the second half, we add and subtract terms as in

$$w_{k+1}^* w_k = w_{k+2}^* w_{k+1} - [w_{k+1}(w_{k+2}^* - w_{k+1}^*) + w_{k+1}^*(w_{k+1} - w_k^*)], \tag{B2}$$

for $k = 1, 3, \ldots, N-3$ only. We obtain

$$\begin{aligned} 4\sum_{k=1,3}^{N-1} w_{k+1}^* w_k &= 2\sum_{k=1,3}^{N-1} w_{k+1}^* w_k + 2\sum_{k=1,3}^{N-3} w_{k+2}^* w_{k+1} \\ &\quad - 2\sum_{k=1,3}^{N-3} [w_{k+1}(w_{k+2}^* - w_{k+1}^*) + w_{k+1}^*(w_{k+1} - w_k^*)] + 2w_N^* w_{N-1}. \end{aligned} \tag{B3}$$

We finally add $-2\sum_{k=1}^{N} w_k^* w_k$. The part of this sum containing odd k's goes together with the first sum above. The even k's up to $N-2$ go with the second sum. We get

$$\begin{aligned} 4\sum_{k=1,3}^{N-1} w_{k+1}^* w_k - 2\sum_{k=1}^{N} w_k^* w_k &= 2\sum_{k=1,3}^{N-1} w_k(w_{k+1}^* - w_k^*) \\ &\quad + 2\sum_{k=1,3}^{N-3} w_{k+1}(w_{k+2}^* - w_{k+1}^*) - 2\sum_{k=1,3}^{N-3} [w_{k+1}(w_{k+2}^* - w_{k+1}^*) \\ &\quad + w_{k+1}^*(w_{k+1} - w_k)] - 2w_N^*(w_N - w_{N-1}). \end{aligned} \tag{B4}$$

The two terms in the second line cancel each other. After incorporating the last term into the sum we get

$$4\sum_{k=1,3}^{N-1} w_{k+1}^* w_k - 2\sum_{k=1}^{N} w_k^* w_k = 2\sum_{k=1,3}^{N-1} [w_k(w_{k+1}^* - w_k^*) - w_{k+1}^*(w_{k+1} - w_k)]. \tag{B5}$$

Appendix C. Calculation of the determinant

The quadratic form in equation (53) is defined by the matrix

$$\begin{pmatrix} \mathrm{i}\tau A_N/\hbar & \mathrm{i}\tau C_N/\hbar + 2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \mathrm{i}\tau C_N/\hbar + 2 & \mathrm{i}\tau B_N/\hbar & -4 & 0 & 4 & 0 & -4 & \cdots \\ 0 & -4 & \mathrm{i}\tau A_{N-1}/\hbar & \mathrm{i}\tau C_{N-1}/\hbar + 2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \mathrm{i}\tau C_{N-1}/\hbar + 2 & \mathrm{i}\tau B_{N-1}\hbar & -4 & 0 & 4 & \cdots \\ 0 & 4 & 0 & -4 & & & & \cdots \\ 0 & 0 & 0 & 0 & & & & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ & & & & & & 0 & 0 \\ & & & & & & -4 & 0 \\ & & & & \cdots & -4 & \mathrm{i}\tau A_1/\hbar & \mathrm{i}\tau C_1/\hbar + 2 \\ & & & & \cdots & 0 & \mathrm{i}\tau C_1/\hbar + 2 & \mathrm{i}\tau B_1/\hbar \end{pmatrix},$$

whose determinant, $\det\tilde{\Delta}_N$, we seek. To simplify the notation we will drop the det symbol in this appendix and use simply $\tilde{\Delta}_N$ for $\det\tilde{\Delta}_N$. It is useful to factor 2i out of each element and call the new determinant Δ_N. Of course,

$$\tilde{\Delta}_N = 2^{2N}\mathrm{i}^{2N}\Delta_N. \tag{C1}$$

This cancels both the 2^N and the sign $(-1)^N$ in equation (53), leaving only Δ_N. Next we do the following sequence of operations that do not change the value of the determinant:

column 2 $\rightarrow$ column 2 + column 4
column 4 $\rightarrow$ column 4 + column 6
$\vdots$
column $N-2 \rightarrow$ column $N-2$ + column $N-4$
line 2 $\rightarrow$ line 2 + line 4
line 4 $\rightarrow$ line 4 + line 6
$\vdots$
line $N-2 \rightarrow$ line $N-2$ + line $N-4$.
This puts the matrix in the block tri-diagonal form:

$$\begin{pmatrix}
\frac{\tau A_N}{2\hbar} & \frac{\tau C_N}{2\hbar}-\mathrm{i} & 0 & 0 & 0 & 0 & 0 & \cdots \\
\frac{\tau C_N}{2\hbar}-\mathrm{i} & \frac{\tau(B_N+B_{N-1})}{2\hbar} & \frac{\tau C_{N-1}}{2\hbar}+\mathrm{i} & \frac{\tau B_{N-1}}{2\hbar} & 0 & 0 & 0 & \cdots \\
0 & \frac{\tau C_{N-1}}{2\hbar}+\mathrm{i} & \frac{\tau A_{N-1}}{2\hbar} & \frac{\tau C_{N-1}}{2\hbar}-\mathrm{i} & 0 & 0 & 0 & \cdots \\
0 & \frac{\tau B_{N-1}}{2\hbar} & \frac{\tau C_{N-1}}{2\hbar}-\mathrm{i} & \frac{\tau(B_{N-1}+B_{N-2})}{2\hbar} & \frac{\tau C_{N-2}}{2\hbar}+\mathrm{i} & \frac{\tau B_{N-2}}{2\hbar} & 0 & \cdots \\
0 & 0 & 0 & \frac{\tau C_{N-2}}{2\hbar}+\mathrm{i} & & & & \cdots \\
0 & 0 & 0 & \frac{\tau B_{N-2}}{2\hbar} & & & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
 & & & & & & 0 & 0 \\
 & & & & & & \frac{\tau C_1}{2\hbar}+\mathrm{i} & \frac{\tau B_1}{2\hbar} \\
 & & & & \cdots & \frac{\tau C_1}{2\hbar}+\mathrm{i} & \frac{\tau A_1}{2\hbar} & \frac{\tau C_1}{2\hbar}-\mathrm{i} \\
 & & & & \cdots & \frac{\tau B_1}{2\hbar} & \frac{\tau C_1}{2\hbar}-\mathrm{i} & \frac{\tau B_1}{2\hbar}
\end{pmatrix}.$$

We can now compute the determinant using Laplace's method. Let Γ_N be the determinant obtained from the matrix above by removing the first line and the first column. The two determinants Δ_N and Γ_N satisfy the following recursion relation:

$$\begin{aligned}
\Delta_N &= \frac{\tau A_N}{2\hbar}\Gamma_N - \left(\frac{\tau C_N}{2\hbar}-\mathrm{i}\right)^2\Delta_{N-1} \\
\Gamma_N &= \frac{\tau(B_N+B_{N-1})}{2\hbar}\Delta_{N-1} - \left(\frac{\tau C_{N-1}}{2\hbar}+\mathrm{i}\right)^2\Gamma_{N-1} + \left(\frac{\tau^2 C_{N-1}^2}{4\hbar^2}+1\right) \\
&\quad\times\frac{\tau B_{N-1}}{2\hbar}\Delta_{N-2} + \frac{\tau B_{N-1}}{2\hbar}\left[1+\frac{\tau^2}{4\hbar^2}\left(C_{N-1}^2 - A_{N-1}B_{N-1}\right)\right]\Delta_{N-2}.
\end{aligned} \tag{C2}$$

Keeping only terms of first order in τ and taking the limit $\tau\rightarrow 0$ we find

$$\begin{aligned}
\frac{\Delta_N-\Delta_{N-1}}{\tau} &= \frac{A_N}{2\hbar}\Gamma_N + \mathrm{i}\frac{C_N}{\hbar}\Delta_{N-1} + \mathcal{O}(\tau^2) \\
\frac{\Gamma_N-\Gamma_{N-1}}{\tau} &= \frac{(B_N+B_{N-1})}{2\hbar}\Delta_{N-1} - \mathrm{i}\frac{C_{N-1}}{\hbar}\Gamma_{N-1} + \frac{B_{N-1}}{\hbar}\Delta_{N-2} + \mathcal{O}(\tau^2)
\end{aligned} \tag{C3}$$

or

$$\dot{\Delta} = \frac{A}{2\hbar}\Gamma + \mathrm{i}\frac{C}{\hbar}\Delta \qquad \dot{\Gamma} = \frac{2B}{\hbar}\Delta - \mathrm{i}\frac{C}{\hbar}\Gamma, \tag{C4}$$

with the initial conditions $\Delta(0) = 1$ and $\Gamma(0) = 0$.

Note that in the case of a harmonic oscillator $H_k = \hbar\omega w_k w_k^*$ and, therefore, $A_k = B_k = 0$ and $C_k = \hbar\omega$. In this case equations (C2) can be solved exactly, without the need to take the continuum limit. We find simply

$$\Delta_N = -\left(\frac{\tau C_N}{2\hbar} - \mathrm{i}\right)^2 \Delta_{N-1} = \left(1 + \frac{\mathrm{i}\omega\tau}{2}\right)^2 \Delta_{N-1}, \tag{C5}$$

which can be iterated to give

$$\Delta_N = \left(1 + \frac{\mathrm{i}\omega\tau}{2}\right)^{2N}. \tag{C6}$$

To solve equations (C4) in the general case we need a last change of variables $\Omega \equiv 2\mathrm{i}\Delta$. In the new variable we get

$$\dot{\Omega} = \mathrm{i}\frac{A}{\hbar}\Gamma + \mathrm{i}\frac{C}{\hbar}\Omega \qquad \dot{\Gamma} = -\mathrm{i}\frac{B}{\hbar}\Omega - \mathrm{i}\frac{C}{\hbar}\Gamma, \tag{C7}$$

with $\Omega(0) = 2\mathrm{i}$ and $\Gamma(0) = 0$. Identifying Γ with u and Ω with v, we recognize these equations immediately as the equations of motion (48) linearized around the stationary trajectory. The solution we seek, $\Delta(T) = \Omega(T)/2\mathrm{i}$, can be obtained with the help of the relations

$$-\mathrm{i}\hbar u'' = \frac{\partial S}{\partial v''} \qquad -\mathrm{i}\hbar v' = \frac{\partial S}{\partial u'}, \tag{C8}$$

where we use a single prime for quantities calculated at $t = 0$ and a double prime when $t = T$. A variation in the second of these equations leads to

$$-\mathrm{i}\hbar\delta v' = \frac{\partial^2 S}{\partial u'^2}\delta u' + \frac{\partial^2 S}{\partial u'\partial v''}\delta v''. \tag{C9}$$

Using $\delta u' = \Gamma(0) = 0$, $\delta v'' = \Omega(T)$ and $\delta v' = \Omega(0) = 2\mathrm{i}$ we get

$$\Omega(T) = 2\mathrm{i}(-\mathrm{i}\hbar)\left(\frac{\partial^2 S}{\partial u'\partial v''}\right)^{-1} \tag{C10}$$

and

$$\Delta = \left(\frac{\mathrm{i}}{\hbar}\frac{\partial^2 S}{\partial u'\partial v''}\right)^{-1}. \tag{C11}$$

References

[1] Klauder J R and Skagerstam B S 1985 *Coherent States: Applications in Physics and Mathematical Physics* (Singapore: World Scientific)

[2] Baranger M, de Aguiar M A M, Keck F, Korsch H J and Schellhaa B 2001 *J. Phys. A: Math. Gen.* **34** 7227

[3] dos Santos L C and de Aguiar M A M 2005 *Braz. J. Phys.* **35** 175

[4] Balazs N L and Jennings B K 1984 *Phys. Rep.* **104** 347

[5] Ozorio de Almeida A M 1998 *Phys. Rep.* **295** 265

[6] Grosche C and Steiner F 1998 *Handbook of Feynman Path Integrals (Springer Tracts in Modern Physics)* (Berlin: Springer)

[7] Hillery M, O'Connel R F, Scully M O and Wigner E P 1984 *Phys. Rep.* **106** 121

[8] Klauder J R 1978 *Continuous Representations and Path Integrals, Revisited (Path Integrals, NATO Advanced Study Institute, Series B: Physics)* ed G J Papadopoulos and J T Devreese (New York: Plenum) p 5

[9] Klauder J R 1979 *Phys. Rev.* D **19** 2349

[10] Klauder J R 1987 *Some Recent Results on Wave Equations, Path Integrals and Semiclassical Approximations (Random Media)* ed G Papanicolaou (Berlin: Springer)
[11] Weissman Y 1982 *J. Chem. Phys.* **76** 4067
[12] Adachi S 1989 *Ann. Phys.* **195** 45
[13] Ribeiro A D, de Aguiar M A M and Baranger M 2004 *Phys. Rev.* E **69** 66204
[14] Pollak E and Shao J 2003 *J. Phys. Chem.* A **107** 7112
[15] Glauber R J 1963 *Phys. Rev.* **131** 2766
[16] Perelomov A M 1972 *Commun. Math. Phys.* **26** 222
[17] Nieto M M and Simmons L M Jr 1979 *Phys. Rev.* D **20** 1321